

# Anomalous Symmetry Fractionalization and Surface Topological Order

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In addition to fractional statistics, anyon excitations of a 2D topologically ordered state can realize symmetry in unusual ways such as carrying fractional quantum numbers, leading to a variety of symmetry enriched topological (SET) phases. While the symmetry fractionalization must be consistent with the fusion and braiding rules of the anyons, not all consistent symmetry fractionalizations can be realized in 2D systems. Instead, certain SETs are anomalous in that they can only occur on the surface of a 3D symmetry protected topological (SPT) phase. In this paper we describe a procedure for identifying an anomalous SET which has a discrete unitary symmetry group  $G$ . The basic idea is to gauge the symmetry and expose the anomaly as an obstruction to defining a consistent topological theory involving both the original anyons and the gauge fluxes. We point out that a class of obstructions are captured by the fourth cohomology group  $H^4(G, U(1))$ , which also labels the set of 3D SPT phases, providing an explicit link to surface topological orders. We illustrate this using the simplest possible example - the projective semion model - where a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry acts on a chiral semion in a way which is only possible on the surface of a 3D SPT phase. Possible extensions to anti-unitary symmetries are discussed, in particular the toric code with Kramers degenerate electric and magnetic charges, is shown to be naturally connected to a 3D SPT phase.

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## I. INTRODUCTION

Recently it has been realized that gapped phases can be distinguished on the basis of symmetry even when that symmetry is unbroken. Short range entangled phases of this form - dubbed ‘symmetry protected topological’ (SPTs) - have been classified using group cohomology<sup>1,2</sup>. However, in two and higher dimensions there are also long range entangled phases supporting fractionalized excitations, and it is an interesting problem to classify these phases in the presence of a symmetry<sup>3-9</sup>. In two dimensions, one approach is to distinguish such ‘symmetry enriched topological’ (SET) phases on the basis of symmetry representation fractionalization on the anyons<sup>3,5</sup>. (Note: this applies when the symmetry doesn’t permute the anyons). For example, a  $\mathbb{Z}_2$  spin liquid with the gauge charge excitations carrying spin 1/2 represents a different SET phase from the one where the gauge charge carries no spin. Indeed, in two dimensions, all possible ways of assigning fractional symmetry quantum numbers to anyons which are compatible with their fusion and braiding rules can be enumerated<sup>5,9</sup>.

However, it is not clear that just because a certain assignment of fractional symmetry representations is compatible with all the fusion and braiding structure, it must necessarily be realizable by a two dimensional Hamiltonian. Previous works have put forward several putative SETs whose assignments are in fact *anomalous*, and incompatible with any 2D symmetric physical realization<sup>10-17</sup>. In these examples, time reversal symmetry is involved and the anomaly is usually exposed by showing that the SET must be chiral when realized in

2D, which necessarily breaks time reversal symmetry unless realized on the surface of a 3D SPT. In this paper, we focus on SETs with unitary discrete symmetries and discuss a general way to detect anomalies in them. We start with the simplest example of this type which we refer to as the ‘projective semion’ theory, where the only quasiparticle in the theory - a semion - carries a projective representation of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry of the system. While such a symmetry representation is compatible with the fusion and braiding structure of the semions, we show that such a theory is anomalous and can never be realized purely in 2D.

With unitary symmetry, one cannot use chiral edge states to identify the anomaly, instead we need a new strategy. The anomaly in the projective semion theory is exposed when we try to gauge the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry. If an SET can be realized in a 2D symmetric model, this gauging process should result in a larger topological theory with more quasiparticles, including the gauge charges and gauge fluxes of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry. However, as we are going to show, such an extension fails for the projective semion theory because braiding and fusion rules cannot be consistently defined for the gauge fluxes and the semion. Therefore, it indicates that the projective semion theory is not possible in 2D.

On the other hand, we show that the projective semion theory can be realized at the surface of a 3D SPT with  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, where symmetry acts anomalously. We discuss in detail the relation between the SPT order in the bulk and the projective semion on the surface.

This projective semion theory represents the simplest example of an anomalous SET theory with unitary and

discrete symmetries. Indeed the method we discuss in the paper can be used to detect anomalies in all SETs with discrete unitary symmetries and provides a direct link between the symmetry anomaly on the surface and the SPT order in the 3D bulk. The mathematics underlying this method of anomaly detection, developed by Etingof et al.<sup>18</sup>, studies the problem of  $G$ -extensions of tensor categories. We use it in this paper to discuss the general case after analyzing in detail the simple example of a projective semion. Satisfyingly, this theory includes a class of obstructions to defining a consistent 2D topological order with both anyons and symmetry fluxes, which are captured by the fourth cohomology group  $H^4(G, U(1))$  of the symmetry group. The same mathematical object is believed to classify all three dimensional SPT phases with unitary symmetry groups<sup>1,2</sup>. Thus the ‘anomalous’ 2D SETs have a natural relation to 3D SPTs, via the surface topological order. Finally, we discuss cases involving time reversal symmetry, and how similar criteria can potentially be applied there as well.

The paper is structured as follows: in section II, we introduce the projective semion model and show that gauging the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry leads to an inconsistency in the braiding and fusion rules of the symmetry fluxes; in section III we present a solvable 3D lattice model that realizes an SPT with a projective semion surface state; in section IV we give a non-linear sigma model representation of the 3D SPT, and discuss the relation between the SPT order in the bulk and the anomaly on the surface; in section V, we discuss generalizations of this method to all SETs with discrete unitary symmetries and propose our conjecture of a general formula that predicts the SPT order in the bulk given the SET state realized on the surface. We close with discussion on future directions including the incorporation of time reversal symmetry into this formalism.

## II. ANOMALY OF THE PROJECTIVE SEMION MODEL IN 2D

### A. The ‘projective semion’ model

The ‘projective semion’ model we consider is a variant of the Kalmeyer-Laughlin chiral spin liquid (CSL)<sup>19</sup>. The Kalmeyer-Laughlin CSL can of course be realized in 2D with the explicit construction given in<sup>19</sup>. However, by slightly modifying the way spin rotation symmetry acts on the semion, we obtain an anomalous theory.

First, recall the setup for the Kalmeyer Laughlin CSL. The degrees of freedom are spin-1/2’s on a lattice. We will not be concerned with the precise form of the Hamiltonian, but only note that it is spin rotation ( $SO(3)$ ) invariant. Thus we can think of the CSL as an  $SO(3)$  symmetry enriched topological (SET) phase. The chiral topological order is the same as the  $\nu = 1/2$  bosonic fractional quantum Hall state which can be described by the

$K = 2$  Chern-Simons gauge theory

$$\mathcal{L} = 2 \frac{\epsilon_{\mu\nu\lambda}}{4\pi} a_\mu \partial_\nu a_\lambda \quad (1)$$

There is one non-trivial quasiparticle, a semion  $s$  which induces a phase factor of  $-1$  when going around another semion. Two semions fuse into the a trivial quasiparticle, which we denote as  $I$ . Moreover each semion carries a spin-1/2 under the  $SO(3)$  symmetry. The CSL is then a nontrivial SET because  $s$  carries a projective representation of  $SO(3)$ .<sup>20</sup> The precise definition of projective representation is given in appendix A. Such an SET theory is of course not anomalous.

In order to get an anomalous theory, reduce the symmetry to the discrete subgroup of 180 degree rotations about the  $x, y$ , and  $z$  axes, which form a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  subgroup of  $SO(3)$ . We denote the group elements as  $g_x, g_y$  and  $g_z$ . The CSL is of course also an SET of this reduced symmetry group. Each semion carries half charge for all three  $\mathbb{Z}_2$  transformations, because 360 degree rotation of a spin-1/2 always results in a phase factor of  $-1$ . Moreover, the three  $\mathbb{Z}_2$  transformations anti-commute with each other and can be represented as

$$\text{CSL: } g_x = i\sigma_x, g_y = i\sigma_y, g_z = i\sigma_z \quad (2)$$

However, now there are other possible SETs, because the semion can now carry extra half integral charges of the  $\mathbb{Z}_2$  symmetries. For example, the semion can carry integral charge under  $g_x$  and  $g_y$  but fractional charge under  $g_z$ . Indeed, we can have three variants of the CSL which we call the ‘projective semion’ models where the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  on the semions can be represented as

$$\begin{aligned} \text{Projective Semion X: } & g_x = i\sigma_x, g_y = \sigma_y, g_z = \sigma_z \\ \text{Projective Semion Y: } & g_x = \sigma_x, g_y = i\sigma_y, g_z = \sigma_z \\ \text{Projective Semion Z: } & g_x = \sigma_x, g_y = \sigma_y, g_z = i\sigma_z \end{aligned} \quad (3)$$

The addition of half charge to the CSL seems completely harmless. First notice that the symmetry representation in the projective semion theories is compatible with the fusion rule of the semion. Two semions fuse into a bosonic particle which is non-fractionalized. Correspondingly, it can be easily checked that having two copies of the same projective representation listed in Eq. 3 gives rise to a trivial representation of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  with integral  $\mathbb{Z}_2$  charges and commuting  $g_x, g_y, g_z$ . On the other hand, topological theories with the semion carrying half charges (in  $\nu = 1/2$  fractional quantum Hall states) and spin-1/2’s (in CSL) have both been identified in explicit models in 2D. However, as we are going to show in the next section, the projective semion theories are anomalous and not possible in purely 2D systems with  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry.

### B. Projective fusion rule of gauge fluxes

The anomaly is exposed when we try to gauge the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry in the projective semion model and

introduce gauge fluxes  $\Omega_x$ ,  $\Omega_y$  and  $\Omega_z$ . The non-trivial projective action of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  on the semion over-constrains the fusion rules of the fluxes, leading to an inconsistency, as we show in this section.

In this section, we will show that:

*The gauge fluxes have a ‘projective’ fusion rule up to a semion.*

which is determined from the way the symmetry acts on the semion.

First, note that each  $\Omega$  actually contains two sectors which differ from each other by a semion. One might try to label one of the sectors as the ‘vacuum’ flux sector  $\Omega$  and the other as the ‘semion’ flux sector  $s\Omega$ , although there is no absolute meaning to which one is which and only the difference between the two sectors matters. Interesting things happen when we consider the fusion rules of the fluxes. Normally, one would expect for example  $\Omega_i \times \Omega_i = I$  ( $I$  denotes the vacuum) and  $\Omega_x \times \Omega_y = \Omega_z$  due to the structure of the symmetry group. However, due to the existence of two sectors we might actually get  $\Omega_i \times \Omega_i = s$  and  $\Omega_x \times \Omega_y = s\Omega_z$ . That is, the gauge fluxes must fuse in the expected way only up to an additional semion.

These ‘projective’ fusion rules can be determined from the action of the symmetry on the semion. Consider for example the Projective Semion X state. One important observation is that bringing a gauge flux around the semion is equivalent to acting with the corresponding symmetry locally on the semion, as shown in Fig.1 (a). Because the semion carries a half charge of  $g_x$ , bringing two  $\Omega_x$ ’s around the semion gives rise to a  $-1$  phase factor, which can be reproduced by bringing another semion around the semion. Therefore, if we imagine fusing the

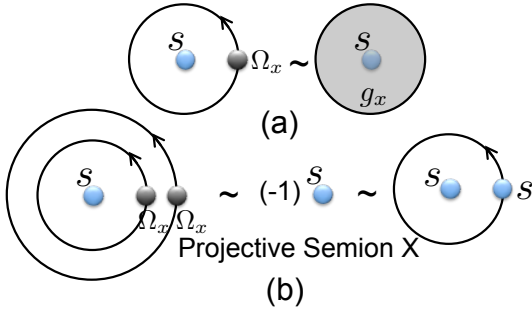


FIG. 1. Fusion rule of gauge flux from symmetry action on the semion: (a) Bringing a gauge flux ( $\Omega_x$ ) around a semion is equivalent to acting with the corresponding symmetry ( $g_x$ ) locally on the semion; (b) Bringing two  $\Omega_x$  gauge fluxes around the semion gives rise to a  $-1$  phase factor, because the semion carries half of the corresponding  $g_x$  charge. This  $-1$  sign can be reproduced by bringing another semion around the original semion.

two  $\Omega_x$  fluxes before bringing them around the semion - a distinction which should not change the global phase factor - we are led to the conclusion that two  $\Omega_x$ ’s fuse

into a semion

$$\Omega_x \times \Omega_x = s \quad (4)$$

Similarly, we find

$$\begin{aligned} \Omega_y \times \Omega_y &= I & \Omega_z \times \Omega_z &= I & \Omega_x \times \Omega_y &= s\Omega_z \\ \Omega_y \times \Omega_x &= \Omega_z & \Omega_y \times \Omega_z &= \Omega_x & \Omega_z \times \Omega_y &= s\Omega_x \\ \Omega_z \times \Omega_x &= s\Omega_y & \Omega_x \times \Omega_z &= \Omega_y \end{aligned} \quad (5)$$

The fusion rules involving the  $s\Omega$  fluxes can be correspondingly obtained by adding  $s$  to both sides. For example,  $s\Omega_x \times \Omega_x = I$ .

In this way, we have established a ‘projective’ fusion rule for the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  gauge fluxes in the Projective Semion X state. The fusion rule is ‘projective’, with a semion coefficient, and can be compactly expressed as a mapping  $\omega$  from two group elements to the semion or the vacuum

$$\begin{aligned} \omega(g_x, g_x) &= s & \omega(g_y, g_y) &= I & \omega(g_z, g_z) &= I \\ \omega(g_x, g_y) &= s & \omega(g_y, g_x) &= I & \omega(g_y, g_x) &= I \\ \omega(g_z, g_y) &= s & \omega(g_z, g_x) &= s & \omega(g_x, g_z) &= I \end{aligned} \quad (6)$$

such that

$$\Omega_g \times \Omega_h = \omega(g, h) \Omega_{gh} \quad (7)$$

The mapping  $\omega$  obeys certain relations. First, consider the fusion of three gauge fluxes  $\Omega_g$ ,  $\Omega_h$  and  $\Omega_k$ . The result should not depend on the order of fusion. We can choose to fuse  $\Omega_g$ ,  $\Omega_h$  together first and then fuse with  $\Omega_k$  or we can choose to fuse  $\Omega_h$ ,  $\Omega_k$  together first and then fuse with  $\Omega_g$ . The equivalence between these two procedures leads to the following relation among the semion coefficients

$$\omega(g, h) \omega(gh, k) = \omega(h, k) \omega(g, hk) \quad (8)$$

It is straight forward to check that this relation is satisfied by the  $\omega$  given in Eq. 6 and we are going to use this relation in our discussion of the next section.

For the other two projective semion states, similar fusion rules can be derived, also with coefficients taking semion values. In fact, in an SET phase (anomalous or not) with discrete unitary symmetries which do not change anyon types, it is generally true that when the symmetry is gauged the gauge fluxes satisfy a projective fusion rule with coefficient in the abelian anyons. We will discuss more about the general situation in section V.

Note that the fusion product of two  $\Omega$ ’s is order dependent. For example,  $\Omega_x \times \Omega_y = s\Omega_z$  while  $\Omega_y \times \Omega_x = \Omega_z$ , which is very different from the usual fusion rules we see in a topological theory. This is because the  $\Omega$ ’s discussed here are not really quasi-particles, but rather the end points of symmetry defect lines introduced in the original SET. Because the  $\Omega$ ’s are all attached to defect lines, they are actually confined. However, we can still define fusion between them and because of the existence of defect lines their fusion is not commutative. If the SET is not anomalous, the  $\Omega$ ’s should first form what is called

a ‘fusion category’, whose properties are discussed in for example Ref. 9 and 21. The fusion product between objects in a fusion category can be non-commutative, but it does have to be associative, and the Pentagon relation<sup>22</sup> illustrated in Fig. 3 must still be satisfied. Moreover, in a non-anomalous SET theory a gauge field can be introduced and the  $\Omega$ ’s can be promoted to deconfined quasi-particles. However, as we shall see in the following, the  $\Omega$ ’s in the projective semion theory do not admit consistent fusion rules which satisfy the pentagon equation, not to mention the promotion to real quasi-particles with extra braiding structures.

It is worth emphasizing at this point that a projective fusion rule for the fluxes is not in itself an indication of a surface anomaly. Indeed, a projective fusion rule for the gauge fluxes exists in many SETs; for example, if we considered the toric code instead of the semion topological order, we could make say the  $\mathbb{Z}_2$  gauge charge carry the sort of fractional  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charges that make the semion theory anomalous, but not have any problem realizing it in a symmetric way in 2D. To expose the anomaly in the projective semion model, we have to do more work.

### C. Anomaly in the statistics of gauge fluxes

We are now ready to see the anomaly from the statistics of the gauge fluxes. In a topological theory with anyonic excitations, the anyon statistics is described by two sets of data: the braiding statistics of exchanging anyon  $a$  with anyon  $b$  and the fusion statistics in the associativity of fusing anyons  $a$ ,  $b$  and  $c$ . The fusion statistics is the phase difference between two processes: the one which fuses  $a$  with  $b$  first before fusing with  $c$  and the one which fuses  $b$ ,  $c$  first before fusing with  $a$ .

For example, in the semion theory discussed here, the only nontrivial braiding statistics happen when two semions are exchanged and this process is shown diagrammatically in Fig. 2 (a). The exchange of two semions leads to a phase factor of  $i$ . Denote the braiding statistics as  $R_{\omega,\omega}$ , where  $\omega = I, s$ .

$$R_{s,s} = i, R_{\omega,\omega} = 1 \text{ otherwise} \quad (9)$$

The only nontrivial fusion statistics happen when three semions are fused together in different orders as shown in Fig. 2 (b). The two orders of fusion differ by a phase factor of  $-1$ . Denote the fusion statistics as  $F_{\omega,\omega,\omega}$ , where  $\omega = I, s$ .

$$F_{s,s,s} = -1, F_{\omega,\omega,\omega} = 1 \text{ otherwise} \quad (10)$$

Similarly, if the symmetry in the projective semion theory can be consistently gauged, then we should be able to define for the gauge fluxes not only the projective fusion rules but also the braiding and fusion statistics involved with exchanging two fluxes or fusing three of them in different orders. These statistics cannot be chosen arbitrarily. In fact, they have to satisfy certain consistency

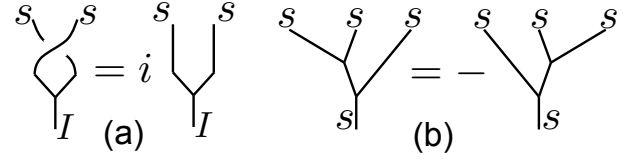


FIG. 2. Braiding and fusion statistics of the semion theory: (a) The exchange of two semions leads to a phase factor of  $i$ ; (b) The two ways of fusing three semions differ by a sign.

conditions, one of which is called the Pentagon equation as shown in Fig. 3.

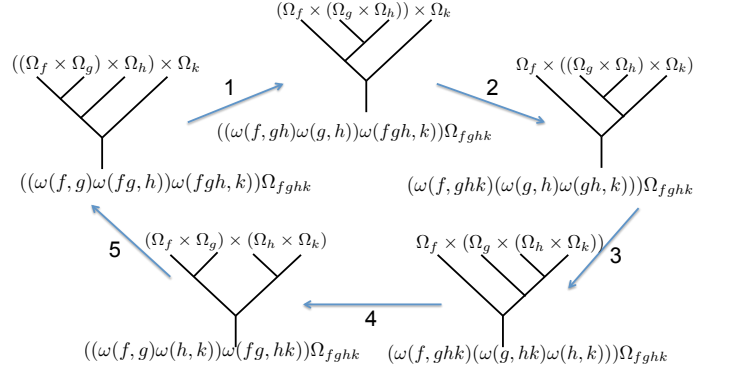


FIG. 3. Pentagon equation for the fusion of four gauge fluxes. Order of fusion is denoted with parenthesis.

The Pentagon equation relates different orders of fusing four gauge fluxes,  $\Omega_f$ ,  $\Omega_g$ ,  $\Omega_h$ , and  $\Omega_k$ . For example, in the figure on the top left corner of Fig. 3,  $\Omega_f$  and  $\Omega_g$  are fused together first, then with  $\Omega_h$  and finally with  $\Omega_k$ . In moving from this figure to the next figure through step 1, we have changed the order (the associativity) in the fusion of the first three fluxes. Such a step is related to a phase factor given by the fusion statistics of the first three gauge fluxes similar to the phase factor shown for three semions in Fig. 2 (b). However, when we go through all configurations in the pentagon and are back to the original configuration, the total phase factor gained should be equal to 1. This is a fundamental requirement for the consistency of the fusion statistics of any anyon theory.

For the gauge fluxes, there is a direct way to check whether this Pentagon equation is satisfied by using the projective fusion rules of the fluxes. By repeated use of the projective fusion rule, the fusion result of the each figure can be reduced to (in fact sophisticated steps are involved in the derivation of the following results, which

we explain in the next section)

$$\begin{aligned}
& ((\omega(f, g)\omega(fg, h))\omega(fgh, k))\Omega_{fghk} \\
& ((\omega(g, h)\omega(f, gh))\omega(fgh, k))\Omega_{fghk} \\
& ((\omega(g, h)\omega(gh, k))\omega(f, ghk))\Omega_{fghk} \\
& ((\omega(h, k)\omega(g, hk))\omega(f, ghk))\Omega_{fghk} \\
& ((\omega(f, g)\omega(h, k))\omega(fg, hk))\Omega_{fghk}
\end{aligned} \tag{11}$$

The meaning of this equation is as follows: each line gives an equivalent representation of the fusion product of the four gauge fluxes  $\Omega_{f,g,h,k}$ , corresponding to one of the five figures in the pentagon diagram. Comparing these, we can see that the difference between the configurations can be characterized by the difference in the semion coefficient, whose braiding and fusion statistics are already known. Therefore, we can derive the phase factor change at each step of the Pentagon using the statistics of the semions. At step 1, there is no extra phase factor involved due to the relation  $\omega(f, g)\omega(fg, h) = \omega(g, h)\omega(f, gh)$  satisfied by the semion coefficients. At step 2, we first need to change the order of fusion such that  $\omega(f, gh)$  is fused with  $\omega(fgh, k)$  first before fusing with  $\omega(g, h)$ ; then using the relation  $\omega(f, gh)\omega(fgh, k) = \omega(gh, k)\omega(f, ghk)$  we relate the second configuration to the third one; finally, we change the order of fusion in the third configuration such that  $\omega(g, h)$  and  $\omega(gh, k)$  are fused first and then with  $\omega(f, ghk)$ . The total phase factor involved in step 2 is then

$$F_{\omega(g,h),\omega(f,gh),\omega(fgh,k)} F_{\omega(g,h),\omega(gh,k),\omega(f,ghk)}^{-1} \tag{12}$$

Similarly, we find that there are extra phase factors at step 4 and 5. At step 5, the total phase factor is

$$F_{\omega(f,g),\omega(h,k),\omega(fg,hk)} F_{\omega(f,g),\omega(fg,h),\omega(fgh,k)}^{-1} \tag{13}$$

Step 4 is a bit more special because it involves the exchange of two semion coefficients: first we change the order of fusion in the fourth configuration such that  $\omega(g, hk)$  and  $\omega(f, ghk)$  are fused first and then with  $\omega(h, k)$ ; then use the relation  $\omega(g, hk)\omega(f, ghk) = \omega(f, g)\omega(fg, hk)$  to relate the fourth configuration to the fifth; change the order of fusion again such that  $\omega(h, k)$  and  $\omega(f, g)$  are fused first before fusing with  $\omega(fg, hk)$ ; finally, exchange  $\omega(h, k)$  and  $\omega(f, g)$  which results in a phase factor of  $R_{\omega(h,k),\omega(f,g)}$ . Therefore, the total phase factor involved is

$$F_{\omega(h,k),\omega(g,hk),\omega(f,ghk)} F_{\omega(h,k),\omega(f,g),\omega(fg,hk)}^{-1} R_{\omega(h,k),\omega(f,g)} \tag{14}$$

Putting all these phase factors together, we should get 1 for all possible choices of  $f, g, h, k$  if a consistent anyon theory can be defined for the gauge fluxes. If not, then there is obstruction to gauging the symmetry, indicating that the SET state is anomalous. Indeed, for the Projective Semion X state defined above, one can directly check that the total phase factor for the four gauge fluxes  $\Omega_z, \Omega_y, \Omega_z, \Omega_y$  is not 1. Therefore, the Projective semion X state is anomalous. Similarly, we can see that the Projective semion Y and Z states are also anomalous.

It is worthwhile to emphasize a slight subtlety at this point: in order to conclude that we have an obstruction, i.e. in order to conclude that there is no possible way to write down consistent fusion and braiding rules for the gauge fluxes and semion, we should in principle check that the pentagon equation cannot be satisfied for *any* choice of defect F-matrices. It turns out that in general the F-matrices of 3 defects are uniquely up to a phase by pentagon equations involving 3 defects and an anyon; in our specific model, these can be chosen (up to said phase) to be equal to 1, something that we implicitly made use of in the above analysis. Now, if the 4-defect pentagon equation fails to be satisfied, we still have the freedom to redefine the 3-defect F matrix by a phase, in the hopes of satisfying the 4-defect pentagon equation. That this is impossible for our projective semion theory is the consequence of a group cohomology calculation, leading to an obstruction in  $H^4(G, U(1))$ , as we discuss in detail in the next section.

#### D. Group cohomology structure of the anomaly

There is actually more structure to the phase factor calculated above than we have discussed. In order to explain this structure, we will make use of group cohomology theory, reviewed in appendix A. Discussion in this section is more formal and mathematical, but given the classification of SPT phases with group cohomology, the identification of the group cohomology structure in the anomaly of the projective semion state allows us to make direct connection to the SPT order in the 3D bulk when we try to realize the projective semion state on the surface of a 3D system. This mathematical structure was identified by Etingof et. al.<sup>18</sup> and we briefly explain the main idea in this section. This general structure applies not only to the projective semion example, but to all SETs with discrete unitary symmetries as well.

First, one can easily see from Eq. 8, using the definition of projective representation, that the gauge fluxes form a projective representation of the symmetry group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  with semion coefficients. Because the semion has a  $\mathbb{Z}_2$  fusion rule, the possible projective fusion rules of the gauge fluxes can be classified by  $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ . Direct calculation shows that  $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ , where the trivial element corresponds to the CSL state and the nontrivial elements correspond to the projective semion X, Y, Z states respectively.

It is generally true that possible projective fusion rules, hence possible symmetry fractionalization patterns, in an SET theory are classified by  $H^2(G, A)$  where  $G$  is the symmetry group and  $A$  denotes the set of abelian anyons<sup>5,9</sup>. Notice that here the coefficients of the projective fusion rule are valued only in the abelian sector of the topological theory, which ensures that the symmetry fractionalization pattern is consistent with the fusion and braiding rules of the topological theory. In more general situations, the anyons can be permuted by the symme-

try in the system. Correspondingly, the  $g$  action on the coefficients  $A$  can be nontrivial, but the  $H^2$  classification as discussed in appendix A still applies.

With the symmetry fractionalization information encoded in  $\omega \in H^2(G, A)$ , we can then move on to determine whether the SET theory is anomalous or not. That is, if we gauge the  $G$  symmetry, whether we can obtain a consistent extended topological theory involving both the gauge charge, gauge fluxes and the original anyons. This is a highly nontrivial process, but fortuitously a mathematical framework has been developed in 18, concerning group extensions of braided categories, which is precisely equipped to deal with these issues. Specifically, this mathematical framework allows us to draw the following important conclusions:

1. In order to determine whether an SET theory is anomalous or not, we only need to look at the fusion and braiding statistics of the original anyons and all the gauge fluxes. If a consistent topological theory can be defined for the original anyons and all the gauge fluxes, then gauge charges can always be incorporated without obstruction.

Therefore, to detect an anomaly in a putative SET theory, we need to look for possible ways to define consistent fusion and braiding statistics for all the gauge fluxes which we denote as  $\Omega$  and the original anyons which we denote as  $\alpha$ . We already know the fusion and braiding statistics of  $\alpha$  which are given by  $R_{\alpha,\alpha}$  and  $F_{\alpha,\alpha,\alpha}$ . (For simplicity of notation we use the same label for the set of anyons in  $\alpha$  and  $\Omega$ . Also we suppress the complication in notation due to nonabelian anyons as long as it does not cause confusion.) What we need to find is the fusion and braiding statistics of  $\Omega$  and those involving both  $\Omega$  and  $\alpha$ :  $R_{\alpha,\Omega}$ ,  $R_{\Omega,\Omega}$ ,  $F_{\alpha,\alpha,\Omega}$ ,  $F_{\alpha,\Omega,\Omega}$  and  $F_{\Omega,\Omega,\Omega}$  (and those with permuted indices).

The strategy is to bootstrap from the known data ( $R_{\alpha,\alpha}$  and  $F_{\alpha,\alpha,\alpha}$ ) and solve for the unknowns using the consistency equations they have to satisfy. The consistency equation comes in two types: the pentagon equation involving the fusion statistics  $F$  and the hexagon equation involving both the braiding and the fusion statistics  $R$  and  $F$ . (For a detailed discussion of these equations see Ref.22.) There is one pentagon equation for every combination of 4 quasiparticles (including both  $\alpha$  and  $\Omega$ ) and there is one hexagon equation for 3 quasiparticles (including both  $\alpha$  and  $\Omega$ ). Of course the pentagon and hexagon equations involving only  $\alpha$  are all satisfied. The next step is to include one  $\Omega$  and try to solve for the  $R_{\alpha,\Omega}$  and  $F_{\alpha,\alpha,\Omega}$  that appear in those equations and so on. If the equations have no solutions, then we have detected an anomaly. Of course, this would be a lengthy process to follow by brute force. Luckily, it has been shown in Ref.18 that there are only two steps at which the equations might have no solutions.

2. The first such step results in an obstruction in  $H^3(G, A)$ , which, when non-zero, signals a lack of any solutions. However, this obstruction appears only when the group  $G$  acts non-trivially on the anyons by permu-

tation, and essentially corresponds to the fact that, when it is non-zero, it is impossible to define an associative fusion product (nevermind requiring the associativity constraint to satisfy the pentagon equation). Since we are only concerned in this paper with a trivial permutation by the group on the set of anyons, we can safely ignore this obstruction.

3. When the first type of obstruction vanishes, there is still a probability that we run into a second type of obstruction when we try to satisfy the pentagon equation of four  $\Omega$ 's, as is the case with the projective semion example. Indeed, the combination of Eq. 12,13,14 together gives the total phase factor up to which the pentagon equation is satisfied:

$$\begin{aligned} \nu(f, g, h, k) = & R_{\omega(h,k),\omega(f,g)} \times \\ & F_{\omega(g,h),\omega(f,gh),\omega(fgh,k)}^{-1} F_{\omega(g,h),\omega(gh,k),\omega(f,ghk)}^{-1} \\ & F_{\omega(f,g),\omega(h,k),\omega(fg,hk)}^{-1} F_{\omega(f,g),\omega(fg,h),\omega(fgh,k)}^{-1} \\ & F_{\omega(h,k),\omega(g,hk),\omega(f,ghk)}^{-1} F_{\omega(h,k),\omega(f,g),\omega(fg,hk)}^{-1} \end{aligned} \quad (15)$$

for  $f, g, h, k \in G$ . It was proved in Ref.18 that the  $\nu(f, g, h, k)$  data forms a four-cocycle of group  $G$  with  $U(1)$  coefficients ( $G$  acts trivially on  $U(1)$ ). Moreover, if we follow the bootstrap steps closely, we see that  $\nu(f, g, h, k)$  does not all have to be identically equal to 1 in order for the pentagon equation to have solutions. This is because the  $F_{\Omega,\Omega,\Omega}$  involved in each step of the pentagon equation (which we took to be the phase difference between different lines in Eq. 11) is actually defined only up to an arbitrary phase factor  $\beta(\Omega, \Omega, \Omega)$ . Therefore, if the  $\nu(f, g, h, k)$  is different from 1 but takes the form:

$$\frac{\beta(f, g, h)\beta(f, gh, k)\beta(g, h, k)}{\beta(fg, h, k)\beta(f, g, hk)} \quad (16)$$

then it can be gauged away. In other words,  $\nu(f, g, h, k)$  has some gauge - or co-boundary - degrees of freedom. Therefore, this type of obstruction is classified by  $H^4(G, U(1))$ , which corresponds exactly to the classification of 3D SPT phases with  $G$  symmetry.

The explanation in this section provides a very brief physical interpretation of some of the results in Ref. 18 (see also 9). Even though this is far from a clear derivation of the result, we extract the most important formula – Eq. 15 – for the identification of the second type of anomaly in an SET theory. For the projective semion theory, we calculated this  $\nu(f, g, h, k)$ , confirmed that it is a four cocycle and moreover checked numerically that it is indeed nontrivial. An analytical proof of this fact is also given in appendix B. We have also checked that  $\nu(f, g, h, k)$  is a trivial four cocycle for the CSL state.

The second thing we want to point out through this discussion is the connection between the SET anomaly and the SPT phases. They are both classified by  $H^4(G, U(1))$  and hence one might expect a close relation between the two, which we will explore further in the remaining sections.



### III. REALIZATION OF THE PROJECTIVE SEMION MODEL ON 3D SURFACE

In the previous section, we established that a system with excitations that have semionic statistics and transform under a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  global symmetry projectively according to Eq. 3 is inconsistent in 2D. In this section, we will demonstrate that such a system *can* be realized at the 2D boundary of a 3D SPT phase. We will do this by presenting an exactly solvable model. In section IV A we will elaborate on this 3D phase by discussing the relation between the projective semion state and  $O(5)$  NLSM description of the SPT bulk. We also present a more physical way to detect the anomaly by threading crossed fluxes in the 3D bulk in section IV B.

Our model is constructed by adding a global symmetry to a Walker-Wang<sup>23</sup>-type lattice model such that its surface admits semionic excitations transforming projectively. Using the Walker-Wang construction, Von Keyserlingk et al.<sup>24</sup> described a 3D loop gas model with deconfined semions living on its surface, and no deconfined excitations in the bulk. In the bulk this model is therefore trivial, while its surfaces are topologically ordered. To make the surface semions transform projectively under a global  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, we will exploit the fact that the ground state of this model is a superposition of closed loop configurations (with relative amplitudes  $\pm 1, \pm i$ ), and that excitations with semionic statistics arise when open strings end on the surface. We will decorate these strings with the equivalent of Haldane chains (for the appropriate projective representation). With this decoration the ground state is necessarily invariant under the global symmetry, as it consists only of closed loops. The end of an open string, however, is also the end of a Haldane chain and hence carries a projective representation. Therefore the semions on the surface, which also occur at the ends of such open strings, transform projectively under the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry.

#### A. The Walker-Wang semion model: a brief review

We begin by briefly reviewing the 3D semion loop gas model. (For a more detailed discussion readers should consult Ref. 24). The Hilbert space consists of a hardcore boson with the 2 states  $n_i = 0, 1$  on each edge of the cubic lattice. The Hamiltonian is a sum of two commuting terms, one acting on vertices in the lattice and one on plaquettes:

$$H = - \sum_V A_V - \sum_P B_P \quad (17)$$

We will choose  $A_V$  and  $B_P$  such that  $[B_{P_i}, B_{P_j}] = [A_{V_i}, A_{V_j}] = [B_{P_i}, A_{V_j}] = 0$  for any pair  $i, j$ . The ground state therefore contains only configurations in which  $A_V$  and  $B_P$  are maximized on all vertices and plaquettes respectively.

Defining

$$\tau_i^z = 1 - 2n_i \quad (18)$$

the vertex operator is:

$$A_V = \prod_{i \in *V} \tau_i^z \quad (19)$$

where  $*V$  is the set of edges entering the vertex  $V$ . This favours configurations in which the number of spin-down edges at each vertex must be even. The ground state is consequently a loop gas (Fig. 4): edges on which  $n_i = 1$  (blue in the Figure) form closed loops if  $B_V = 1$  at each vertex.

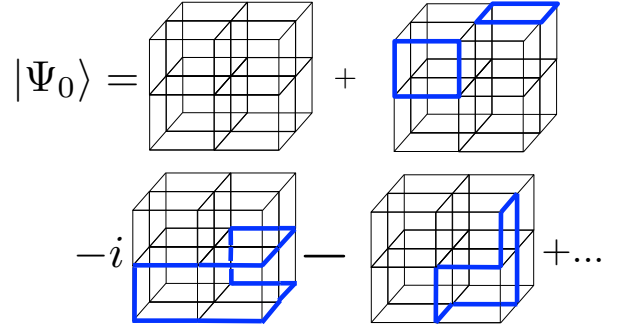


FIG. 4. Some configurations in the semion ground state. The vertex condition ensures that only configurations with an even number of edges on which  $n_i = 1$  (shown in blue in the Figure) can meet at a vertex, such that the ground state is a superposition of loops. The relative amplitudes of these loop configurations are given by the phase factor (21).

The plaquette term  $B_P$  simultaneously flips  $\tau^z$  on all edges around a plaquette, and assigns a configuration-dependent phase to the result:

$$B_P = - \left( \prod_{i \in \partial P} \tau_i^x \right) \Theta(P) \Phi_{O,O'} \Phi_{U,U'} \quad (20)$$

where  $\partial P$  is the set of all edges bordering  $P$ . Exactly as in the 3D Toric code<sup>25,26</sup> model, the product over  $\tau^x$  ensures that the ground state is a superposition of closed loops. Their relative coefficients are determined by the second, third, and fourth terms in Eq. (20), which are phase factors that depend only on  $\{n_i\}$  on edges on or adjacent to the plaquette  $P$ . We will choose the phase factors  $\Theta(P), \Phi_{O,O'}, \Phi_{U,U'}$  such that the coefficients of different loop configurations in the ground state of the loop gas are given (up to an overall global phase) by

$$C_{\{L\}} = Z_{\{L\}}^{CS} \quad (21)$$

where  $Z_{\{L\}}^{CS}$  is the (Euclidean) partition function of a 3D  $U(1)_2$  Chern-Simons theory. Specifically, every linking of loops incurs an extra factor of  $-1$ , while every counter-clockwise (clockwise) twist in a loop induces a factor of  $i$

$(-i)$ . There is also a phase factor of  $(-1)^{N_L}$ , where  $N_L$  is the number of loops.

To obtain the coefficient of  $(-1)^{N_L}$ , we choose:

$$\Theta(P) = \left( \prod_{j \in {}^*P} i^{n_j} \right) \prod_{\{i,j\} \in \{L\text{-pairs}\}} (-1)^{n_i n_j} \quad (22)$$

where  ${}^*P$  denotes the set of all edges entering the plaquette  $P$ . Essentially, this term ensures that if  $\prod_{i \in \partial P} \tau_i^x$  changes the number of loops in a particular loop configuration, the action of  $B_P$  on this configuration is negative (discounting, for the moment, the effect of  $\Phi$ ). However, since 6 edges meet at a vertex, at vertices where  $n_i = 1$  on four or more edges, there can be ambiguity in determining the number of loops in a given configuration. This can be resolved by “point-splitting” each vertex in the cubic lattice into three trivalent vertices, following Walker and Wang<sup>23</sup> (Fig. 5 b). The L-pairs in Eq. (20) (shown in green in the Figure) are chosen such that in the ground state each loop configuration appears with a relative phase factor of  $(-1)^{N_L}$ , where  $N_L$  is the number of closed loops on this point-split lattice: the factors of  $(-1)^{n_i n_j}$  explicitly cancel the factors of  $(i)^2$  that can occur if two edges that are in  ${}^*P$  on the cubic lattice, but are *not* in  ${}^*P$  after we have point-split the vertices, have  $n_i = 1$ .

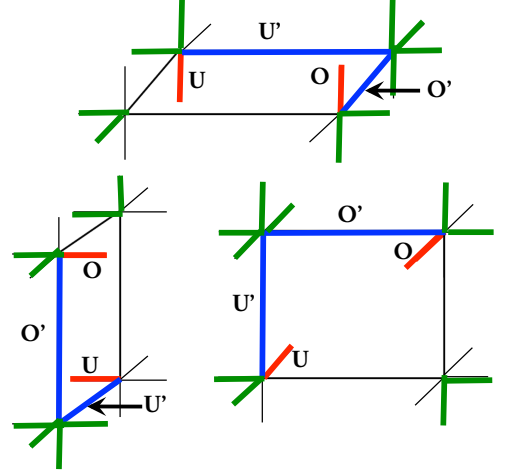
To obtain the phases appropriate to  $Z_{\{L\}}^{CS}$ , we choose:

$$\Phi_{O,O'} = i^{n_O(1+2n_{O'})}, \quad \Phi_{U,U'} = i^{n_U(1+2n_{U'})} \quad (23)$$

To define this term, we must fix a particular 2D projection of the cubic lattice. The two edges  $O$  (for “over”) and  $U$  (for “under”) are edges that are projected into the plaquette from above and below, respectively, in this projection.  $O'$  and  $U'$  are neighbouring edges in  $\partial P$ , as shown in Fig. 5.  $\Phi$  ensures that if flipping the spins in  $\partial P$  introduces a clockwise (counter-clockwise) twist in a loop, the resulting configuration will appear in the ground state with a relative phase of  $i$  ( $-i$ ). (It is also responsible for the extra factor of  $(-1)$  that arises if the action of  $B_P$  links a pair of previously unlinked loops).

Though not immediately apparent, it can be shown that with this choice of  $\Theta, \Phi$ ,  $[B_{P_i}, B_{P_j}] = 0$  for all pairs of plaquettes. Hence the entire spectrum of our model Hamiltonian can be determined exactly. One might guess that this spectrum contains two distinct types of defects: pairs of “vertex” defects, for which  $A_{V_1} = A_{V_2} = -1$  at the affected vertices, correspond to adding an open string to the loop gas connecting vertices  $V_1$  and  $V_2$ ; and “plaquette” defects, in which the eigenvalues of the plaquette term are  $B_P = -1$  along a closed string of plaquettes.<sup>27</sup> This guess is correct if we choose  $\Theta = \Phi = 1$ ; however, for the choices given above it turns out that in the bulk of our model, a pair of vertex defects is always connected by a line of plaquette defects.<sup>24</sup> Hence open strings in the bulk cost a finite energy per unit length, and vertex defects are confined.

(a)



(b)

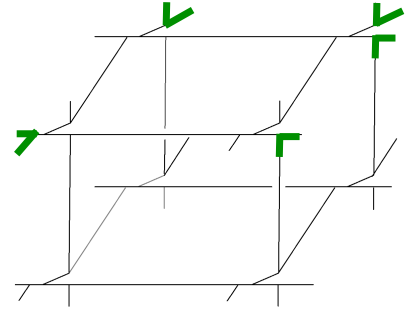


FIG. 5. (a) The model is defined on the cubic lattice, with a fixed angle of projection. In this angle, there are two edges ( $O$  and  $U$ ) that are projected into each plaquette from above and below, respectively, shown in red. These (and their partner edges  $O'$  and  $U'$ , shown here in blue) are used to define certain phase factors in the plaquette operator. There are also special “L-pairs” of edges which are important to determining the overall phase factor associated with a plaquette. Edges involved these L-pairs are shown in green. At vertices with only two green edges, there is a single L-pair between these two edges. At vertices with all four green links, all possible pairs should be taken. (b) Separating the hexavalent vertices of the cubic lattice into trivalent vertices resolves any ambiguity about the total number of loops in a given configuration, and hence about the sign with which this configuration appears in the ground state. Walker and Wang<sup>23</sup> proposed the point-splitting shown here for the cubic lattice. Our choice of L-pairs (shown here for the upper horizontal plaquette in green) assigns phases to loop configurations as appropriate to this particular point splitting.

On the surface, however, vertex defects are deconfined: if  $V_1$  and  $V_2$  are vertices on the boundary of our 3D system, there exist operators that create states in which the eigenvalues obey  $A_{V_1} = A_{V_2} = -1$ , but  $B_P = 1$  everywhere. The corresponding excited states will consist of superpositions of closed loops, together with an open



string connecting the vertices  $V_1$  and  $V_2$ . Further, these vertex defects are semions: exchanging them multiplies the excited-state wave function by a phase factor of  $\pm i$  (Fig. 6). Both the bulk and surface spectrum are derived in detail in Ref. 24.

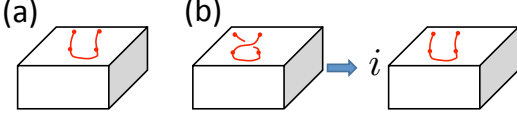


FIG. 6. Beginning with the configuration shown in (a), we exchanging two open string ends at the surface. Returning the strings to their original configuration (b) multiplies the eigenstate of the Walker-Wang model by  $\pm i$ , demonstrating that these defects have semionic statistics.

### B. Decorating the model with a global symmetry

We next show how to decorate this model such that the surface semion transforms projectively under a global  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry. Though in practise the model is an SPT only if the semion transforms projectively according to Eq. (3), we first discuss the case where the semion carries spin-1/2, as in the Kalmeyer-Laughlin spin liquid<sup>19</sup>, in which the loops in our loop gas ground state are true spin-1/2 Haldane chains. To obtain an SPT, we will replace the full  $SU(2)$  symmetry with a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, and take our spins to transform via one of the possibilities listed in Eq. (3).

We begin by enlarging our Hilbert space by adding six auxillary degrees of freedom at each vertex (one associated with each edge). Each auxillary degree of freedom can be either a spin 1/2 (transforming projectively under  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ) or a spinless particle  $b$  (transforming in the singlet representation of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ). We will also impose the constraint that, within this enlarged Hilbert space, only states with an even number of spins at each vertex are allowed. This ensures that the vertex degree of freedom transforms in a linear representation of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

Within this enlarged Hilbert space, it is convenient to define the modified edge variables

$$|\tilde{0}\rangle = |0\rangle|b_i b_j\rangle, \quad |\tilde{1}\rangle = |1\rangle(|\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle)/\sqrt{2} \quad (24)$$

where  $\uparrow, \downarrow$  represent the physical spin states,  $b$  are the spinless states, and  $i, j$  index the auxillary variables associated with the two ends of the edge.  $|\tilde{1}\rangle$  describes an edge in which a hard-core boson occurs in conjunction with two spin degrees of freedom, which combine to form a singlet.  $|\tilde{0}\rangle$  is an edge with no boson, and where the auxillary degrees of freedom are both spinless.

We can express our Hamiltonian in terms of the decorated spin operators:

$$\tilde{\tau}_i^z = |\tilde{0}_i\rangle\langle\tilde{0}_i| - |\tilde{1}_i\rangle\langle\tilde{1}_i|, \quad \tilde{\tau}_i^x = |\tilde{0}_i\rangle\langle\tilde{1}_i| + |\tilde{1}_i\rangle\langle\tilde{0}_i| \quad (25)$$

These act like Pauli spin operators within the Hilbert space (24): by construction,  $(\tilde{\tau}^x)^2 = (\tilde{\tau}^z)^2 = 1$ , and  $\tilde{\tau}^x \tilde{\tau}^z = -\tilde{\tau}^z \tilde{\tau}^x$ . Importantly, they act only on the set of auxillary vertex variables associated with the edge in question, so that decorated spin operators acting on adjacent edges commute. Note also that  $\tilde{\tau}^x$  conserves the total spin, since it replaces a singlet-bonded pair of spin-1/2's at adjacent vertices with a pair of spin 0 particles.

The Hamiltonian is:

$$H = H_0 - \sum_V \tilde{A}_V - \sum_P \tilde{B}_P \quad (26)$$

where  $\tilde{A}_V, \tilde{B}_P$  are as in Eq.'s (20, 19), with  $\tau_i^{x,z}$  replaced by  $\tilde{\tau}_i^{x,z}$ , and  $n_i$  by  $\tilde{n}_i = \frac{1}{2}(1 - \tilde{\tau}_i^z)$ .  $H_0$  is a potential term favouring the edge states  $|\tilde{0}\rangle$  and  $|\tilde{1}\rangle$ :

$$H_0 = -|\tilde{0}\rangle\langle\tilde{0}| - |\tilde{1}\rangle\langle\tilde{1}| \quad (27)$$

Let us now discuss the spectrum of this new Hamiltonian. By construction, any configuration in which all loops are closed will have lowest energy if all edges are in one of the two states  $|\tilde{0}\rangle, |\tilde{1}\rangle$ . This is possible when  $\prod \tau^z = 1$  about each vertex: in this case each hard-core boson can have an associated spin 1/2 at the vertex without violating the constraint; the possibilities are shown in Fig.7(a). At vertices where  $\prod \tau^z = -1$ , it is not possible for all edges to be in one of the two states  $|\tilde{0}\rangle, |\tilde{1}\rangle$  without violating the constraint, as shown in Fig.7(b). Hence whenever a string ends, the constraint that the total number of spins at each vertex must be even ensures that an unpaired spin 1/2 remains at or near the vertex. Moving this unpaired spin away from the vertex in question produces a series of edges along which  $H_0$  is not minimized (Fig. 8). In other words, the unpaired spin is confined by a linear energy penalty to be close to the violated vertex. Therefore an unpaired spin 1/2 is bound (by a linear confining potential) to each end-point of an open string.

Defects in the original semion model involving only plaquette violations are unaffected by the decoration. This includes closed vortex loops and the open flux loop connecting a pair of confined charges. In particular, semions on the surface remain deconfined. Because they are also bound by a confining potential to a spin-1/2, these deconfined surface semions transform projectively under the global symmetry, exactly as in the Kalmeyer-Laughlin spin liquid.

We note in passing that in the presence of open strings, the plaquette term as defined here will be violated at the string endpoints, since there are necessarily edges that are neither in state  $|\tilde{0}\rangle$  or state  $|\tilde{1}\rangle$ . This affects the energy of the semions at the end-points of these open strings, but not their statistics or transformations under symmetry.

The construction given so far realizes a system with a Kalmeyer-Laughlin spin liquid state on its surface, by associating an open Haldane chain with the semion excitation. Our main objective, however, is to obtain surface states that are not possible in 2 dimensions. We can do

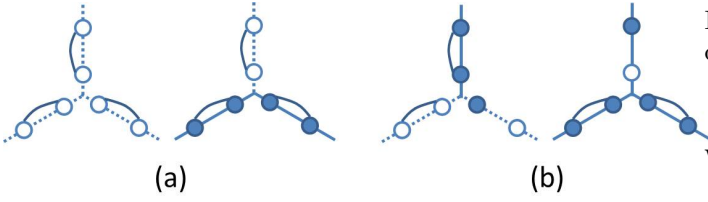


FIG. 7. (a) low energy and (b) high energy configurations at each vertex. Dotted lines represent links without strings while solid lines represent links with strings. Empty circles represent spinless vertex particles while solid circles represent spin-1/2's. Two connected circles form a singlet.

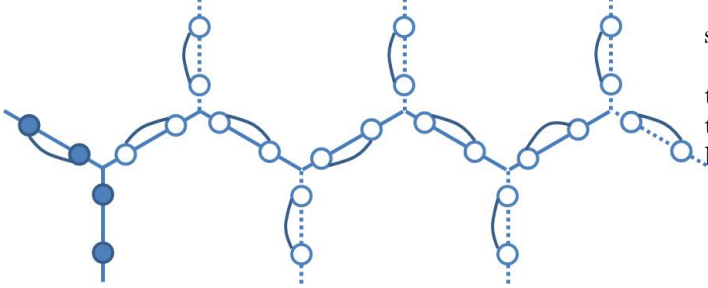


FIG. 8. Possible configuration where no isolated spin 1/2's appear near the end of strings.

this with a very minor modification of the construction given above: it suffices to take the spin 1/2 degrees of freedom to transform under  $\mathbb{Z}_2 \times \mathbb{Z}_2$  according to Eq. 3. In this case our spins are not really spins since we have assumed that  $SU(2)$  spin rotation invariance is broken; however a spin singlet will still transform as a net singlet under symmetry. Hence the operator  $\tilde{\sigma}^x$  still interchanges two different trivial representations, and hence does not violate the symmetry.

In Appendix C, we describe how to generalize this construction to obtain other models with surface anyons that transform projectively under an appropriate global symmetry group.

#### IV. THE 3D SPT

We now turn our attention to understanding the 3D SPT suggested by the model constructed in the previous section. We will first describe it using an  $O(5)$  NLSM; we then describe a way to insert fluxes of the global symmetry that reveal the surface anomaly.

##### A. Obtaining the symmetric semion surface state from the $O(5)$ NLSM construction of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

An alternative approach to constructing SPTs is to use non-linear sigma models with a theta term. In particular,

Ref.10 and 28 argue that a wide class of 3d SPTs can be described by the following action:

$$S_{3d} = \int d^3x d\tau \frac{1}{g} (\partial_\mu \vec{n})^2 + S_\theta$$

where

$$S_\theta = \frac{2\pi i}{64\pi^2} \int d^3x d\tau \epsilon_{abcde} \epsilon^{\mu\nu\rho\delta} n^a \partial_\mu n^b \partial_\nu n^c \partial_\rho n^d \partial_\delta n^e$$

Specifically, for  $G = \mathbb{Z}_2^A \times \mathbb{Z}_2^B$ , Ref.28 argues that the following action of  $G$  realizes a non-trivial  $G$ -SPT:

$$\mathbb{Z}_2^A : n_{1,2} \rightarrow -n_{1,2}, n_a \rightarrow n_a (a = 3, 4, 5)$$

$$\mathbb{Z}_2^B : n_1 \rightarrow n_1, n_a \rightarrow -n_a (a = 2, 3, 4, 5)$$

We will see how to construct one of our anomalous semion surface states as a symmetric termination of this 3D SPT.

Before we do this, let us first consider an alternative action of this symmetry group, which actually corresponds to a trivial 3D SPT. For clarity, we will denote this group by  $\tilde{\mathbb{Z}}_2^A \times \tilde{\mathbb{Z}}_2^B$ . Its action is:

$$\tilde{\mathbb{Z}}_2^A : n_{1,2} \rightarrow -n_{1,2}, n_a \rightarrow n_a (a = 3, 4, 5)$$

$$\tilde{\mathbb{Z}}_2^B : n_{2,3} \rightarrow -n_{2,3}, n_a \rightarrow -n_a (a = 1, 4, 5)$$

Now, this is simply the subgroup of 180 degree rotations inside the  $SO(3)$  that rotates  $n_1, n_2, n_3$ , and since we know that there is no non-trivial 3D SPT of  $SO(3)$  in 3D, the corresponding  $\tilde{\mathbb{Z}}_2^A \times \tilde{\mathbb{Z}}_2^B$  SPT must be trivial as well. But let us examine its surface anyway. To do this, consider the effective action for the complex field  $n_4 + in_5$  on the 2D surface, after having integrated out  $n_{1,2,3}$ . According to the arguments of Ref.10 and 28, the theta term ensures that a single vortex of  $n_4 + in_5$  carries a spin 1/2. Now, we can add fluctuations and proliferate doubled vortices bound to single charges - i.e. we think of this as a system with  $U(1)$  charge conservation symmetry that acts on  $n_4 + in_5$ , and drive it to a  $\nu = 1/2$  bosonic Laughlin state, while maintaining the  $SO(3)$  symmetry that acts on  $n_1, n_2, n_3$ . We have then constructed a chiral spin liquid at the surface - the non-trivial quasiparticle is a semion, and because it descends from a single vortex, it carries spin 1/2. The chiral spin liquid is of course a perfectly valid 2D symmetry enriched phase, which we expect since the bulk  $SO(3)$  SPT is trivial.

Having understood this trivial case, we turn to  $G = \mathbb{Z}_2^A \times \mathbb{Z}_2^B$  defined above, which corresponds to the non-trivial SPT. The analysis is in fact completely the same, with the only difference being that  $\mathbb{Z}_2^B$  differs from  $\tilde{\mathbb{Z}}_2^B$  by the transformation  $n_{4,5} \rightarrow -n_{4,5}$ , i.e. a  $\pi$  rotation of the  $U(1)$ :  $n_4 + in_5 \rightarrow -n_4 - in_5$ . Once we drive the system into the  $\nu = 1/2$  bosonic Laughlin state, this extra  $\pi$  rotation causes  $\mathbb{Z}_2^B$  to pick up an extra phase factor, equal to  $\pi$  times the charge of the semion (which is 1/2), i.e.  $e^{i\pi/2} = i$ . In other words, the semion now carries an extra half charge of  $\mathbb{Z}_2^B$ , compared to the case of the chiral spin liquid. This is precisely our anomalous surface state. The other anomalous surface states can be constructed by changing the roles of the generators of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

### B. Crossed $\mathbb{Z}_2$ fluxes argument

Although the irreparable inconsistency in the pentagon equations for the fluxes already implies that our semion surface state cannot be symmetrically realized in 2D, it is nice to have a more physically tangible manifestation of this anomaly. Indeed, one such manifestation is the following: if we put such a putative surface state on a  $T^2$  torus geometry, then inserting a flux of  $\mathbb{Z}_2^A$  in one cycle and a flux of  $\mathbb{Z}_2^B$  in the other results in a half charge of  $\mathbb{Z}_2^B$  appearing on the torus surface, as shown in Fig. 9. Clearly this cannot happen in a symmetric model in 2D, where the degrees of freedom all form ordinary linear representations of the symmetry group. Another half charge appears along the flux line that pierces the bulk, well separated from the surface.

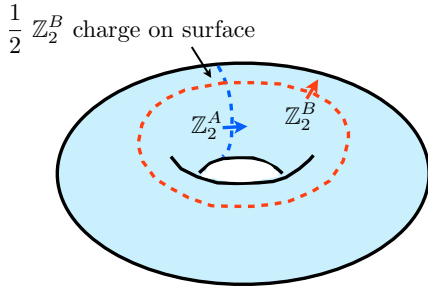


FIG. 9. If the projective semion state is put on a  $T^2$  torus geometry, then inserting a flux of  $\mathbb{Z}_2^A$  in one cycle and a flux of  $\mathbb{Z}_2^B$  in the other results in a half charge of  $\mathbb{Z}_2^B$  appearing on the torus surface, which is not possible for a purely 2D symmetric system.

Although we believe that this linked fluxes argument can be phrased entirely in 2D, we will instead demonstrate it in a 3D context, where the  $T^2$  forms a surface of a 3D bulk. It is actually easiest to see this in the decorated domain wall picture of Ref.29. Take the following geometry as shown in Fig. 10: an  $xy$  plane with a hole (taken to be the unit disk centered at the origin), and then thickened in the  $z$  direction, with periodic boundary conditions imposed in the  $z$  direction. Just to be specific, let's take  $z$  in the interval  $[0, 1]$  and periodically identify  $z = 0$  and  $z = 1$ . This defines a 3D bulk whose surface is a 2D torus: namely, the unit circle in the  $xy$  plane times the  $[0, 1]$  interval in the  $z$  direction. This is equivalent to the thickened torus geometry mentioned initially, except that the other surface of the torus is out at infinity.

Now put a flux of  $\mathbb{Z}_2^A$  through the  $z$  cycle. This just means we insert a defect surface (domain wall) of  $\mathbb{Z}_2^A$  on the  $xy$  plane, at  $z = 0$ . According to the arguments of Ref.29, this domain wall is just a 2D SPT (in the  $xy$  plane) of  $\mathbb{Z}_2^B$ . So inserting a flux of  $\mathbb{Z}_2^B$  through the hole - which is just the second cycle of our torus - brings a half charge of  $\mathbb{Z}_2^B$  to the surface, as desired.

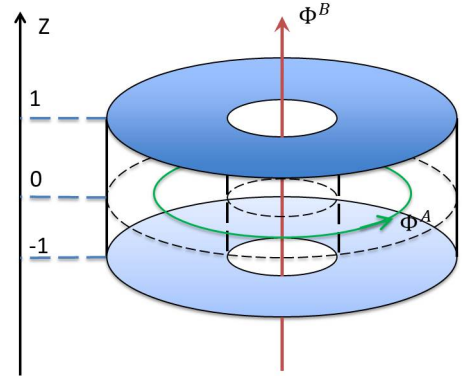


FIG. 10. Inserting crossed fluxes of  $\mathbb{Z}_2^A$  and  $\mathbb{Z}_2^B$  respectively through the 3D bulk of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  SPT phase. The top and bottom surfaces are identified and the middle dashed plane at  $z = 0$  is a  $\mathbb{Z}_2^A$  defect plane and holds a 2D SPT state of  $\mathbb{Z}_2^B$ .

Alternatively, we can consider the same geometry in the sigma model framework. Once again, put a flux of  $\mathbb{Z}_2^A$  through the  $z$  cycle. Then the trick is to integrate out  $n_1$  to get a 2D  $O(4)$  sigma model with theta term for  $n_2, n_3, n_4, n_5$ . The original action of  $\mathbb{Z}_2^B$ , namely  $n_i \rightarrow -n_i$ , descends to this  $O(4)$  sigma model, and we recover the sigma model for the non-trivial 2D SPT of  $\mathbb{Z}_2^B$ , as in eq. 46 of Ref.28.

### V. SUMMARY

In this paper, we studied in detail a symmetry enriched topological state - the projective semion state - which cannot be realized in 2D symmetric models even though the fractional symmetry action on the anyons is consistent with all the fusion and braiding rules of the chiral semion topological order. The anomaly is exposed when we try to gauge the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry and fail to find a solution for the fusion statistics of the gauge fluxes which satisfies the pentagon equation. On the other hand, we demonstrate that the projective semion state can be realized on the surface of a 3D system and that there is a close connection between the surface SET order and the symmetry protected topological order in the bulk.

The projective semion state is the simplest example of an anomalous SET state with discrete unitary symmetries. Our discussion in section II illustrates a general procedure for detecting anomalies in a large class of such SET states. In particular, we want to emphasize two points:

1. The fractional symmetry action on the anyons gives rise to a projective fusion rule of the gauge fluxes once the symmetry is gauged, with the abelian anyon in the original topological theory as the co-efficient.

2. For SETs with discrete unitary symmetries, when gauging the symmetry, the possible obstruction comes in two types: the  $H^3$  type and the  $H^4$  type. Eq. 15 gives the general formula to detect the  $H^4$  type. If the  $\nu(f, g, h, k)$  in Eq. 15 forms a nontrivial four cocycle of  $G$  with  $U(1)$  coefficients, then the SET is anomalous.

Thus, not only can an anomalous SET be unambiguously identified, it can also be related to a particular 3D SPT phase when it is of the  $H^4$  type.

## VI. DISCUSSION AND FUTURE DIRECTIONS

Thus far we have focused on discrete unitary symmetry groups  $G$  and identifying whether a given SET is anomalous. However we mention the following two extensions which we conjecture to be true:

1. Anomalous SET with  $H^4$  type obstruction (vanishing  $H^3$  type obstruction) can be realized on the surface of a 3D SPT phase characterized by the same four cocycle  $\nu \in H^4(G, U(1))$ .
2. Eq. 15 applies to SETs with time reversal symmetry as well.

That is, there should be a constructive procedure to realize a particular anomalous SET, given an element of  $H^4$ , which is the surface topological order of the corresponding 3D bosonic SPT phase, at least for discrete unitary  $G$ . (A 3D SPT with no consistent surface topological order was recently discussed<sup>30</sup>, although that involved bulk fermions and a combination of continuous symmetry and time reversal).

For time reversal symmetry, although it is currently not known how to introduce time reversal fluxes at the same level as for a unitary symmetry, we conjecture that the same formula Eq. 15 works in detecting the  $H^4$  type obstruction. This comes from the observation of the following example: the  $\mathbb{Z}_2$  gauge theory with both the gauge charge  $e$  and gauge flux  $m$  transforming as  $T^2 = -1$  (sometimes called eTmT). It is believed that this state is anomalous and appears on the surface of the 3D SPT with time reversal symmetry<sup>10</sup> which is described by the cohomology classification, in particular the nontrivial element in  $H^4(\mathbb{Z}_2^T, U(1))$ , where the time reversal symmetry acts nontrivially on the  $U(1)$  coefficients by taking complex conjugation. Suppose that we can define in some notion a time reversal gauge flux. Then the projective fusion rule would be

$$\Omega_T \times \Omega_T = f \quad (28)$$

where  $f$  is the bound particle of  $e$  and  $m$ . Equivalently, we can write

$$\omega(T, T) = f, \omega = I \text{ otherwise} \quad (29)$$

This reflects the fractional symmetry action where  $T^2 = -1$  on  $e$  and  $m$  because  $f$  has a  $-1$  braiding statistics with both  $e$  and  $m$ . Now we can use this information to calculate  $\nu(f, g, h, k)$  in Eq. 15. The  $\mathbb{Z}_2$  gauge theory has trivial  $F$ . Therefore, Eq. 15 reduces to

$$\nu(f, g, h, k) = R_{\omega(h, k), \omega(f, g)} \quad (30)$$

which is nontrivial only when  $f = g = h = k = T$  and  $\nu(T, T, T, T) = -1$ . This is exactly the nontrivial 4 cocycle of time reversal<sup>2</sup>.

Even though at this moment, we are not sure what gauging time reversal means in general, recent work indicates that this notion can be formalized<sup>31,32</sup>. From this particular example, we expect that the procedure and result discussed in Ref.18 might be generalized to treat anti-unitary symmetries as well.

## VII. ACKNOWLEDGMENT

When we were writing up the paper, we learned of other works on anomalous SET's with unitary discrete symmetries<sup>32,33</sup>. We are very grateful to helpful discussions with Meng Cheng, Senthil Todadri, Ryan Thorngren, Alexei Kitaev, and Netanel Lindner. XC is supported by the Miller Institute for Basic Research in Science at UC Berkeley, AV is supported by NSF DMR 0645691.

### Appendix A: Projective representation and group cohomology

The following definition works only for unitary symmetries because we are not dealing with time reversal symmetry in this paper.

Matrices  $u(g)$  form a projective representation of symmetry group  $G$  if

$$u(g_1)u(g_2) = \omega(g_1, g_2)u(g_1g_2), \quad g_1, g_2 \in G. \quad (A1)$$

Here  $\omega(g_1, g_2) \in U(1)$  and  $\omega(g_1, g_2) \neq 1$ , which is called the factor system of the projective representation. The factor system satisfies

$$\omega(g_2, g_3)\omega(g_1, g_2g_3) = \omega(g_1, g_2)\omega(g_1g_2, g_3), \quad (A2)$$

for all  $g_1, g_2, g_3 \in G$ . If  $\omega(g_1, g_2) = 1, \forall g_1, g_2$ , this reduces to the usual linear representation of  $G$ .

A different choice of pre-factor for the representation matrices  $u'(g) = \beta(g)u(g)$  will lead to a different factor system  $\omega'(g_1, g_2)$ :

$$\omega'(g_1, g_2) = \frac{\beta(g_1)\beta(g_2)}{\beta(g_1g_2)}\omega(g_1, g_2). \quad (A3)$$

We regard  $u'(g)$  and  $u(g)$  that differ only by a pre-factor as equivalent projective representations and the corresponding factor systems  $\omega'(g_1, g_2)$  and  $\omega(g_1, g_2)$  as belonging to the same class  $\omega$ .

Suppose that we have one projective representation  $u_1(g)$  with factor system  $\omega_1(g_1, g_2)$  of class  $\omega_1$  and another  $u_2(g)$  with factor system  $\omega_2(g_1, g_2)$  of class  $\omega_2$ , obviously  $u_1(g) \otimes u_2(g)$  is a projective presentation with factor system  $\omega_1(g_1, g_2)\omega_2(g_1, g_2)$ . The corresponding class  $\omega$  can be written as a sum  $\omega_1 + \omega_2$ . Under such an addition rule, the equivalence classes of factor systems form an Abelian group, which is called the second cohomology group of  $G$  and is denoted as  $H^2(G, U(1))$ . The identity element  $1 \in H^2(G, U(1))$  is the class that corresponds to the linear representation of the group.

The above discussion on the factor system of a projective representation can be generalized which gives rise to a cohomology theory of groups.

For a group  $G$ , let  $M$  be a  $G$ -module, which is an abelian group (with multiplication operation) on which  $G$  acts compatibly with the multiplication operation (i.e. the abelian group structure) on  $M$ :

$$g \cdot (ab) = (g \cdot a)(g \cdot b), \quad g \in G, \quad a, b \in M. \quad (\text{A4})$$

For example,  $M$  can be the  $U(1)$  group and  $a$  a  $U(1)$  phase. The multiplication operation  $ab$  is then the usual multiplication of the  $U(1)$  phases. The group action is trivial  $g \cdot a = a$  for unitary symmetries considered here. Or  $M$  can be a  $\mathbb{Z}_2$  group and  $a$  is the semion or the vacuum sector in the  $K = 2$  Chern-Simons theory. The multiplication  $ab$  is then the fusion between anyons. The group action  $g \cdot a = b$  encodes how the anyon sectors get permuted under the symmetry, which is trivial for the projective semion example discussed in this paper but can be nontrivial in general.

Let  $\omega_n(g_1, \dots, g_n)$  be a function of  $n$  group elements whose value is in the  $G$ -module  $M$ . In other words,  $\omega_n : G^n \rightarrow M$ . Let  $C^n(G, M) = \{\omega_n\}$  be the space of all such functions. Note that  $C^n(G, M)$  is an Abelian group under the function multiplication  $\omega_n''(g_1, \dots, g_n) = \omega_n(g_1, \dots, g_n)\omega_n'(g_1, \dots, g_n)$ . We define a map  $d_n$  from  $C^n(G, U(1))$  to  $C^{n+1}(G, U(1))$ :

$$(d_n \omega_n)(g_1, \dots, g_{n+1}) = g_1 \cdot \omega_n(g_2, \dots, g_{n+1}) \omega_n^{(-1)^{n+1}}(g_1, \dots, g_n) \times \prod_{i=1}^n \omega_n^{(-1)^i}(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \quad (\text{A5})$$

Let

$$B^n(G, M) = \{\omega_n | \omega_n = d_{n-1} \omega_{n-1} | \omega_{n-1} \in C^{n-1}(G, M)\} \quad (\text{A6})$$

and

$$Z^n(G, M) = \{\omega_n | d_n \omega_n = 1, \omega_n \in C^n(G, M)\} \quad (\text{A7})$$

$B^n(G, M)$  and  $Z^n(G, M)$  are also Abelian groups which satisfy  $B^n(G, M) \subset Z^n(G, M)$  where  $B^1(G, M) \equiv \{1\}$ .  $Z^n(G, M)$  is the group of  $n$ -cocycles and  $B^n(G, M)$  is the

group of  $n$ -coboundaries. The  $n$ th cohomology group of  $G$  is defined as

$$H^n(G, M) = Z^n(G, M) / B^n(G, M) \quad (\text{A8})$$

When  $n = 1$ , we find that  $\omega_1(g)$  satisfies

$$\omega_1(g_1)\omega_1(g_2) = \omega_1(g_1 g_2) \quad (\text{A9})$$

Therefore, the 1st cocycles of a group with  $U(1)$  coefficient are the one dimensional representations of the group.

Moreover, we can check that the consistency and equivalence conditions (Eq. A2 and A3) of factor systems of projective representations are exactly the cocycle and coboundary conditions of 2nd cohomology group. Therefore, 2nd cocycles of a group with  $U(1)$  coefficient are the factor systems of the projective representations of the group. Similarly, we can check that if we use the semion / vacuum sector as a  $\mathbb{Z}_2$  coefficient, then the projective fusion rule of the gauge fluxes discussed in section II B is a 2nd cocycle of the symmetry group with  $\mathbb{Z}_2$  coefficient.

When  $n = 3$ , from

$$(d_3 \omega_3)(g_1, g_2, g_3, g_4) = \frac{\omega_3(g_2, g_3, g_4)\omega_3(g_1, g_2 g_3, g_4)\omega_3(g_1, g_2, g_3)}{\omega_3(g_1 g_2, g_3, g_4)\omega_3(g_1, g_2, g_3 g_4)} \quad (\text{A10})$$

we see that

$$Z^3(G, U(1)) = \{\omega_3 | \frac{\omega_3(g_2, g_3, g_4)\omega_3(g_1, g_2 g_3, g_4)\omega_3(g_1, g_2, g_3)}{\omega_3(g_1 g_2, g_3, g_4)\omega_3(g_1, g_2, g_3 g_4)} = 1\}. \quad (\text{A11})$$

and

$$B^3(G, U(1)) = \{\omega_3 | \omega_3(g_1, g_2, g_3) = \frac{\omega_2(g_2, g_3)\omega_2(g_1, g_2 g_3)}{\omega_2(g_1 g_2, g_3)\omega_2(g_1, g_2)}\}, \quad (\text{A12})$$

which give us the third cohomology group  $H^3(G, U(1)) = Z^3(G, U(1)) / B^3(G, U(1))$ .

Similarly, when  $n = 4$ , from

$$(d_4 \omega_4)(g_1, g_2, g_3, g_4, g_5) = \frac{\omega_4(g_2, g_3, g_4, g_5)\omega_4(g_1, g_2 g_3, g_4, g_5)\omega_4(g_1, g_2, g_3, g_4 g_5)}{\omega_4(g_1 g_2, g_3, g_4, g_5)\omega_4(g_1, g_2, g_3 g_4, g_5)\omega_4(g_1, g_2, g_3, g_4)} \quad (\text{A13})$$

we see that

$$Z^4(G, U(1)) = \{\omega_4 | \frac{\omega_4(g_2, g_3, g_4, g_5)\omega_4(g_1, g_2 g_3, g_4, g_5)\omega_4(g_1, g_2, g_3, g_4 g_5)}{\omega_4(g_1 g_2, g_3, g_4, g_5)\omega_4(g_1, g_2, g_3 g_4, g_5)\omega_4(g_1, g_2, g_3, g_4)} = 1\}. \quad (\text{A14})$$

and

$$B^4(G, U(1)) = \{\omega_4 | \omega_4(g_1, g_2, g_3, g_4) = \frac{\omega_3(g_2, g_3, g_4)\omega_3(g_1, g_2 g_3, g_4)\omega_3(g_1, g_2, g_3)}{\omega_3(g_1 g_2, g_3, g_4)\omega_3(g_1, g_2, g_3 g_4)}\}, \quad (\text{A15})$$

which give us the third cohomology group  $H^4(G, U(1)) = Z^4(G, U(1)) / B^4(G, U(1))$ .

## Appendix B: Showing that the 4-cocycle we get is non-trivial

Although we have numerically verified that the 4-cocycle we obtained from the pentagon anomaly argument corresponds to a non-trivial cohomology class in  $H^4(G, U(1))$ , it is nice to also have an analytic proof of this fact. This is what we demonstrate in this appendix. The notation is:  $[\nu]$  is the  $H^4(G, U(1))$  obstruction class, and  $[\omega_1], [\omega_2]$  are  $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$  classes. The former is computed from the latter, and in fact Ref.18 show that:

$$[\nu]([\omega_1] + [\omega_2]) = [\nu]([\omega_1]) [\nu]([\omega_2]) b([\omega_1], [\omega_2])$$

where the “bilinear form”  $b$  is defined by taking the full braid of  $\omega_1(f, g)$  with  $\omega_2(g, h)$ . For our purposes  $[\omega_1]$  will correspond to the chiral spin liquid (which is not anomalous), and  $[\omega_2]$  will correspond to a theory where the semion only binds a half-charge of some  $\mathbb{Z}_2 \subset \mathbb{Z}_2 \times \mathbb{Z}_2$ . It is useful to let  $u_1$  and  $u_2$  denote elements of  $H^1(\mathbb{Z}_2, \mathbb{Z}_2)$  which correspond to the first and second  $\mathbb{Z}_2$ . Then (using multiplication in the cohomology ring) we can have  $[\omega_2] = u_1^2, u_2^2$ , or  $u_1^2 + u_2^2$ , whereas  $[\omega_1]$  will be denoted  $v$ , the class of the chiral spin liquid. We know that both the CSL and the theory corresponding to  $[\omega_2]$  are realizable in 2d, so their obstructions vanish:  $[\nu](v) = [\nu](u_1^2) = [\nu](u_2^2) = 1$ . So we see that the obstruction corresponding to  $v + \alpha_1 u_1^2 + \alpha_2 u_2^2$  is simply  $(b(v, u_1^2))^{\alpha_1} (b(v, u_2^2))^{\alpha_2}$ , where  $\alpha_i = 0, 1$ . Now, in the case of our semion theory, the full braid of two anyons gives  $-1$  only when both anyons are equal to  $s$ ; if we view the anyons  $\mathbb{Z}_2 = \{0, 1\}$  where  $0$  is the trivial anyon and  $1$  is  $s$ , then this is just multiplication in the field  $\mathbb{Z}_2$ . Hence  $b(v, u_i^2)$  is simply the product of  $u_i^2$  and  $v$  in the cohomology ring with  $\mathbb{Z}_2$  coefficients. This is  $u_1^3 u_2$  in the case  $i = 1$  and  $u_1 u_2^3$  in the case  $i = 2$ . In particular, both of these are nonzero when viewed as elements of  $H^4(G, \mathbb{Z}_2)$ : this is because, according to the Kunneth formula,  $H^4(G, \mathbb{Z}_2)$  is spanned as a  $\mathbb{Z}_2$  vector space by  $\{u_1^4, u_1^3 u_2, u_1^2 u_2^2, u_1 u_2^3, u_2^4\}$ .

The only thing we now need to check is that both  $u_1^3 u_2$  and  $u_1 u_2^3$  are nonzero as  $H^4(G, U(1))$  cohomology classes. In fact,  $H^4(G, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ , and the kernel of the natural map  $H^4(G, \mathbb{Z}_2) \rightarrow H^4(G, U(1))$  is precisely  $\{u_1^4, u_1^2 u_2^2, u_2^4\}$ . To prove this, first note that the map  $H^4(G, \mathbb{Z}_2) \rightarrow H^4(G, U(1))$  is equivalent to the coboundary map  $H^4(G, \mathbb{Z}_2) \rightarrow H^5(G, \mathbb{Z})$  from the long exact sequence associated to  $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2$ , under the natural identification of  $H^4(G, U(1))$  and  $H^5(G, \mathbb{Z})$ . Then we use the explicit Kunneth formula for cohomology with  $\mathbb{Z}$  coefficients, which contains both the “normal” part that is already visible with  $\mathbb{Z}_2$  coefficients, and the torsion part in one degree higher, and the fact that the torsion part maps into the normal part under the coboundary map, to conclude that the kernel of  $H^4(G, \mathbb{Z}_2) \rightarrow H^4(G, U(1))$  is precisely  $\{u_1^4, u_1^2 u_2^2, u_2^4\}$ .

## Appendix C: Generalization of the Walker-Wang model to other cases

The Walker-Wang semion model discussed in section III is only one member of a family of models that have no deconfined excitations in the bulk, but are topologically ordered on the surface<sup>23,24</sup>. We can use a construction analogous to the one presented above to decorate these models with global symmetries such that the surface anyons transform projectively.

A more general Walker-Wang model consists of an  $n$ -state system on each edge of the (point-split) cubic lattice. In the models of interest to us, each of these  $n$  states can be associated with an anyon type that will appear at the surface. In general, to fix the state on a particular edge we must also fix an orientation of this edge; the label  $a$  can then be viewed as a flux along the edge in question, whose direction is specified by this orientation. The flux in the opposite direction is given by the conjugate anyon label  $\bar{a}$ .

The Walker-Wang Hamiltonian has the general form (17), with commuting vertex and plaquette operators. The plaquette term is discussed in detail in Refs. 23 and 24; for our purposes it is sufficient to know that (provided our anyon model is modular) this plaquette operator confines all open strings in the bulk, but allows deconfined anyons at the surface. To specify the vertex operator, we must use the fusion rules of the anyon model, which dictate how anyons proximate in space can combine to give new anyons (or possibly the trivial state  $|0\rangle$ , with no anyons at all). The vertices  $V$  that are allowed in the ground state are those where the anyons entering  $V$  can be fused to each other to give  $|0\rangle$ . For example, a vertex with  $a$  entering on one edge,  $\bar{a}$  entering (or equivalently,  $a$  exiting) on another edge, and all other edges in the state  $|0\rangle$  is always allowed. (For the 2-state model described above, this is the only allowed vertex). However, in general there will be other allowed vertex types as well.

The simplest example of this is a  $\mathbb{Z}_3$  model in which  $\bar{0} = 0, \bar{1} = 2, \bar{2} = 1$ , and  $1$  fuses with  $2$  to give  $0$ . The vertices allowed in the ground state (on the point-split lattice, where all vertices are trivalent) are ones in which a  $1$  (or  $2$ ) anyon flux enters the vertex on one edge, and exits on another, or vertices at which  $3$  anyons fluxes of the same type all enter.

How do we decorate this model such that each anyon transforms in a particular representation of our global symmetry group at the surface? We first enlarge the Hilbert space at each (trivalent) vertex by including  $3$  auxillary sites (one associated to each edge entering the vertex). The Hilbert space of each auxillary site contains a set of objects transforming in different representations of  $G$ . In our example above, this set contained an object  $b$  transforming in the singlet representation, and a spin  $1/2$  (transforming, unsurprisingly, in the spin- $1/2$  representation). More generally, let  $r_a$  be the representation that we wish to make the anyon  $a$  transform in. Then



our auxillary Hilbert space must contain an object transforming in  $r_a$  for every  $a$ . For notational simplicity, we will call this object  $r_a$  from now on.

In order for our construction to work, we must impose the condition that our choice of  $r_a$  is consistent with the fusion rules of the anyon model. Specifically, we will require that  $r_0$  is the trivial representation, and that conjugate anyon types transform under conjugate representations in the symmetry group:

$$\bar{r}_a = r_{\bar{a}} \quad (\text{C1})$$

We also require that

$$a \times b = c \quad \Rightarrow \quad r_a \times r_b = r_c \times \text{linear reps} \quad (\text{C2})$$

(For non-abelian anyons,  $c$  may contain more than one anyon type. However, we will require the projective part of the symmetry action to be the same for all possible products of fusion.)

The next step is to construct analogues of the states  $|\tilde{0}\rangle$  and  $|\tilde{1}\rangle$  for this more general case. Because the edges are oriented, it is natural to favour configurations in which the edge starting at vertex  $V_1$  and ending at vertex  $V_2$ , with anyon label  $a$ , has auxillary variables in  $r_a$  at  $V_1$  and in  $\bar{r}_a$  at  $V_2$ . The analogue of the two states  $|1\rangle$  and  $|0\rangle$  above are states in which an edge with anyon label  $a$  has these two auxillary variables combining to the trivial representation (i.e., in which  $r_a$  and  $\bar{r}_a$  are in a “singlet” state):

$$|\tilde{a}\rangle = |a\rangle |P_{r_a \times \bar{r}_a = r_0}\rangle \quad (\text{C3})$$

We may favour such configurations energetically by introducing a potential term

$$H_0 = \sum_{e=\text{edges}} \sum_{a=1}^n |\tilde{a}_e\rangle \langle \tilde{a}_e| \quad (\text{C4})$$

We next impose the constraint that only linear representations are allowed at vertices. In the ground state, the tensor product of the representations associated with the anyon types that meet at each vertex contains the identity, so Eq. (C2) ensures that this constraint is compatible with minimizing  $H_0$  on each edge. At vertices where the net anyon flux is not zero, however, this constraint forces us to include at least one edge on which  $H_0$  is violated. Let the anyon types of the 3 edges (all oriented into the vertex) be  $a, b$  and  $c$ , and let them be fused at the vertex to create a fourth anyon  $d$ . Let us choose the edge with label  $a$  to be an excited state of  $H_0$ . Then edges  $b$  and  $c$  contribute  $\bar{r}_b \times \bar{r}_c$  to the vertex; hence (up to a tensor product with linear representations, which is not important for our purposes) the auxillary variable associated with  $a$  must carry a representation of  $r_b \times r_c$ . It follows that the edge  $a$  has two auxillary variables, which together transform in the representation  $r_a \times r_b \times r_c$ . (Recall that the other end of our  $a$ -labelled edge had better carry  $r_a$ , to avoid having edges at adjacent vertices that are also in excited states). But we have required that (up to a tensor product with linear representations),  $r_a \times r_b \times r_c = r_d$ . Hence a vertex with net anyon flux  $d$  necessarily has the correct projective component of its symmetry transformation. (In cases where this leaves the representation under which  $d$  transforms ambiguous, it is possible to add additional potential terms to remove this ambiguity).

Evidently, with this construction it is possible to define operators that mix the  $|\tilde{a}\rangle$  variables in a manner consistent with the symmetry, since for all states  $|\tilde{a}\rangle$  the edge carries a trivial representation of the symmetry. We may thus as above construct the plaquette term of the Walker-Wang Hamiltonian in the  $\tilde{a}$  variables to obtain a model in which anyons are confined in the bulk, and transform in the desired projective representations on the surface.

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